Lecture 31

Equivalence Theorems, Huygens' Principle

Electromagnetic equivalence theorems are useful for simplifying solutions to many problems. Also, they offer physical insight into the behaviour of electromagnetic fields of a Maxwellian system. They are closely related to the uniqueness theorem and Huygens' principle. One application is their use in studying the radiation from an aperture antenna or from the output of a lasing cavity. These theorems are discussed in many textbooks [31, 45, 49, 60, 167]. Some authors also call it Love's equivalence principles [168] and credit has been given to Schelkunoff as well [169].

You may have heard of another equivalence theorem in special relativity. It was postulated by Einstein to explain why light ray should bend around a star. The equivalence theorem in special relativity is very different from those in electromagnetics. One thing they have in common is that they are all derived by using Gedanken experiment (thought experiment), involving no math. But in this lecture, we will show that electromagnetic equivalence theorems are also derivable using mathematics, albeit with more work.

31.1 Equivalence Theorems or Equivalence Principles

In this lecture, we will consider three equivalence theorems: (1) The inside out case. (2) The outside in case. (3) The general case. We will derive these theorems using thought experiments or Gedanken experiments. As shall be shown later, they can also be derived mathematically using Green's theorem.

31.1.1 Inside-Out Case

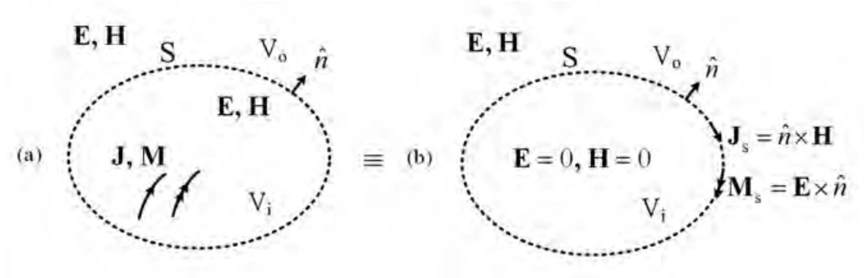

Figure 31.1: The inside-out problem where equivalence currents are impressed on the surface S to produce the same fields outside in V_o in both cases.

In this case, we let **J** and **M** be the time-harmonic radiating sources inside a surface S radiating into a region $V = V_o \cup V_i$. They produce **E** and **H** everywhere. We can construct an equivalence problem by first constructing an imaginary surface S. In this equivalence problem, the fields outside S in V_o are the same in both (a) and (b). But in (b), the fields inside S in V_i are zero.

Apparently, the tangential components of the fields are discontinuous at S. This is not possible for a Maxwellian fields unless surface currents are impressed on the surface S. We have learned from electromagnetic boundary conditions that electromagnetic fields are discontinuous across a current sheet. Then we ask ourselves what surface currents are needed on surface S so that the boundary conditions for field discontinuities are satisfied on S. Clearly, surface currents needed for these field discontinuities are to be impressed on S are

$$\mathbf{J}_s = \hat{n} \times \mathbf{H}, \qquad \mathbf{M}_s = \mathbf{E} \times \hat{n} \tag{31.1.1}$$

We can convince ourselves that $\hat{n} \times \mathbf{H}$ and $\mathbf{E} \times \hat{n}$ just outside S in both cases are the same. Furthermore, we are persuaded that the above is a bona fide solution to Maxwell's equations.

- The boundary conditions on the surface S satisfy the boundary conditions required of Maxwell's equations.

- By the uniqueness theorem, only the equality of one of them $\mathbf{E} \times \hat{n}$, or $\hat{n} \times \mathbf{H}$ ons S, will guarantee that **E** and **H** outside S are the same in both cases (a) and (b).

The fact that these equivalence currents generate zero fields inside S is known as the extinction theorem. This equivalence theorem can also be proved mathematically, as shall be shown.

31.1.2 Outside-in Case

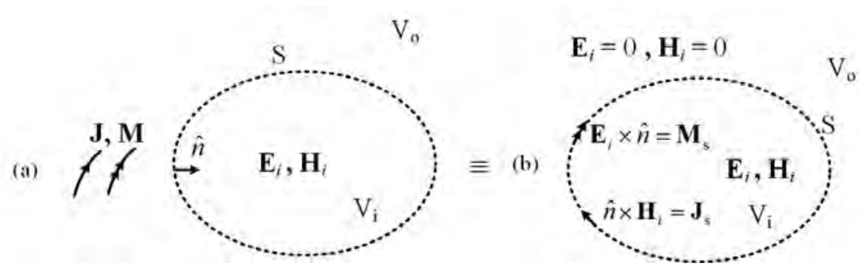

Figure 31.2: The outside-in problem where equivalence currents are impressed on the surface S to produce the same fields inside in both cases.

Similar to before, we find an equivalence problem (b) where the fields inside S in V_i is the same as in (a), but the fields outside S in V_o in the equivalence problem is zero. The fields are discontinuous across the surface S, and hence, impressed surface currents are needed to account for these discontinuities.

Then by the uniqueness theorem,[1] the fields $\mathbf{E}_i$, $\mathbf{H}_i$ inside V in both cases are the same. Again, by the extinction theorem, the fields produced by $\mathbf{E}_i \times \hat{n}$ and $\hat{n} \times \mathbf{H}_i$ are zero outside S.

31.1.3 General Case

From these two cases, we can create a rich variety of equivalence problems. By linear superposition of the inside-out problem, and the outside-in problem, then a third equivalence problem is shown in Figure 31.3:

[1] We can add infinitesimal loss to ensure that uniqueness theorem is satisfied in this enclosed volume V_i.

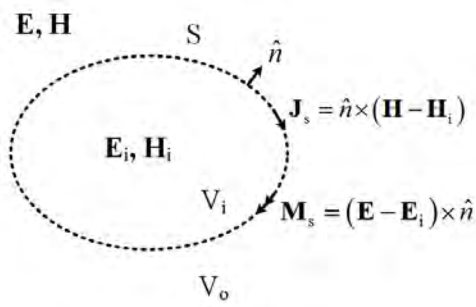

Figure 31.3: The general case where the fields are non-zero both inside and outside the surface S. Equivalence currents are needed on the surface S to support the discontinuities in the fields.

31.2 Electric Current on a PEC

Using the equivalence problems in the previous section, we can derive other corollaries of equivalence theorems. We shall show them next.

First, from reciprocity theorem, it is quite easy to prove that an impressed current on the PEC cannot radiate. We can start with the inside-out equivalence problem. Then using a Gedanken experiment, since the fields inside S is zero for the inside-out problem, one can insert an PEC object inside S without disturbing the fields $\mathbf{E}$ and $\mathbf{H}$ outside. As the PEC object grows to snugly fit the surface S, then the electric current $\mathbf{J}_s = \hat{n} \times \mathbf{H}$ does not radiate by reciprocity. Only one of the two currents is radiating, namely, the magnetic current $\mathbf{M}_s = \mathbf{E} \times \hat{n}$ is radiating, and $\mathbf{J}_s$ in Figure 31.4 can be removed. This is commensurate with the uniqueness theorem that only the knowledge of $\mathbf{E} \times \hat{n}$ is needed to uniquely determine the fields outside S.

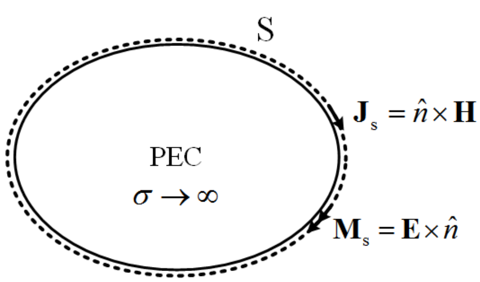

Figure 31.4: On a PEC surface, only one of the two currents is needed since an electric current does not radiate on a PEC surface.

31.3 Magnetic Current on a PMC

Again, from reciprocity, an impressed magnetic current on a PMC cannot radiate. By the same token, we can perform the Gedanken experiment as before by inserting a PMC object inside S. It will not alter the fields outside S, as the fields inside S is zero. As the PMC object grows to snugly fit the surface S, only the electric current $\mathbf{J}_s = \hat{n} \times \mathbf{H}$ radiates, and the magnetic current $\mathbf{M}_s = \mathbf{E} \times \hat{n}$ does not radiate and it can be removed. This is again commensurate with the uniqueness theorem that only the knowledge of the $\hat{n} \times \mathbf{H}$ is needed to uniquely determine the fields outside S.

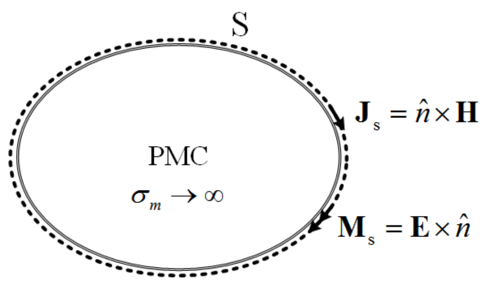

Figure 31.5: Similarly, on a PMC surface, only an electric current is needed to produce the field outside the surface S.

31.4 Huygens' Principle and Green's Theorem

Huygens' principle shows how a wave field on a surface determines the wave field outside the surface S. This concept was expressed by Huygens heuristically in the 1600s [170]. But the mathematical expression of this idea was due to George Green[2] in the 1800s. This concept can be expressed mathematically for both scalar and vector waves. The derivation for the vector wave case is homomorphic to the scalar wave case. But the algebra in the scalar wave case is much simpler. Therefore, we shall discuss the scalar wave case first, followed by the electromagnetic vector wave case.

[2]George Green (1793-1841) was self educated and the son of a miller in Nottingham, England [171].

31.4.1 Scalar Waves Case

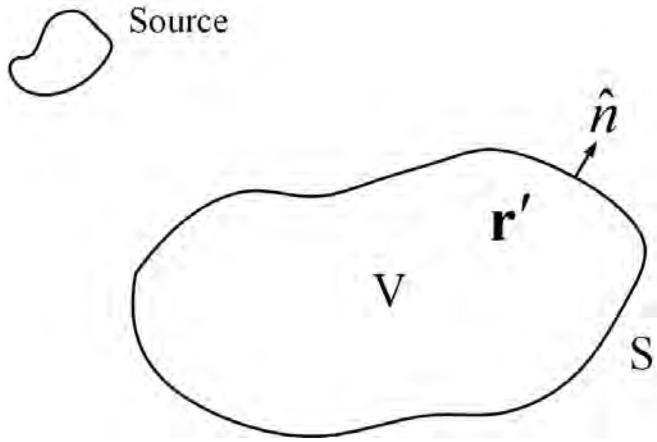

Figure 31.6: The geometry for deriving Huygens' principle for scalar wave equation.

For a $\psi(\mathbf{r})$ that satisfies the scalar wave equation

$$(\nabla^2 + k^2)\,\psi(\mathbf{r}) = 0, \tag{31.4.1}$$

the corresponding scalar Green's function $g(\mathbf{r}, \mathbf{r}')$ satisfies

$$(\nabla^2 + k^2)\,g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'). \tag{31.4.2}$$

Next, we multiply (31.4.1) by $g(\mathbf{r}, \mathbf{r}')$ and (31.4.2) by $\psi(\mathbf{r})$. And then, we subtract the resultant equations and integrating over a volume V as shown in Figure 31.6. There are two cases to consider: when $\mathbf{r}'$ is in V, or when $\mathbf{r}'$ is outside V. Thus, we have

$$\left.\begin{array}{ll} \text{if } \mathbf{r}' \in V, & \psi(\mathbf{r}') \\ \text{if } \mathbf{r}' \notin V, & 0 \end{array}\right\} = \int_V d\mathbf{r}\,[g(\mathbf{r}, \mathbf{r}')\nabla^2\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla^2 g(\mathbf{r}, \mathbf{r}')], \tag{31.4.3}$$

The left-hand side evaluates to different values depending on where $\mathbf{r}'$ is due to the sifting property of the delta function $\delta(\mathbf{r} - \mathbf{r}')$. Since $g\nabla^2\psi - \psi\nabla^2 g = \nabla\cdot(g\nabla\psi - \psi\nabla g)$, the left-hand side of (31.4.3) can be rewritten using Gauss' divergence theorem, giving[3]

$$\left.\begin{array}{ll} \text{if } \mathbf{r}' \in V, & \psi(\mathbf{r}') \\ \text{if } \mathbf{r}' \notin V, & 0 \end{array}\right\} = \oint_S dS\,\hat{n}\cdot[g(\mathbf{r}, \mathbf{r}')\nabla\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla g(\mathbf{r}, \mathbf{r}')], \tag{31.4.4}$$

where S is the surface bounding V. The above is the Green's theorem, or the mathematical expression that once $\psi(\mathbf{r})$ and $\hat{n}\cdot\nabla\psi(\mathbf{r})$ are known on S, then $\psi(\mathbf{r}')$ away from S could be

[3]The equivalence of the volume integral in (31.4.3) to the surface integral in (31.4.4) is also known as Green's theorem [82].

found. This is similar to the expression of equivalence principle where $\hat{n} \cdot \nabla\psi(\mathbf{r})$ and $\psi(\mathbf{r})$ are equivalence sources on the surface S. The first term on the right-hand side radiates via the Green's function $g(\mathbf{r}, \mathbf{r}')$ while the second term radiates via the normal derivative of the Green's function, namely $\nabla g(\mathbf{r}, \mathbf{r}')$. Since the derivative of a Green's function yields a dipole field, the second term corresponds to sources that radiate like dipoles pointing normally to the surface S. In acoustics, these are known as single layer and double layer (or dipole layer) sources, respectively. The above mathematical expression also embodies the extinction theorem that says if $\mathbf{r}'$ is outside V, the left-hand side evaluates to zero.

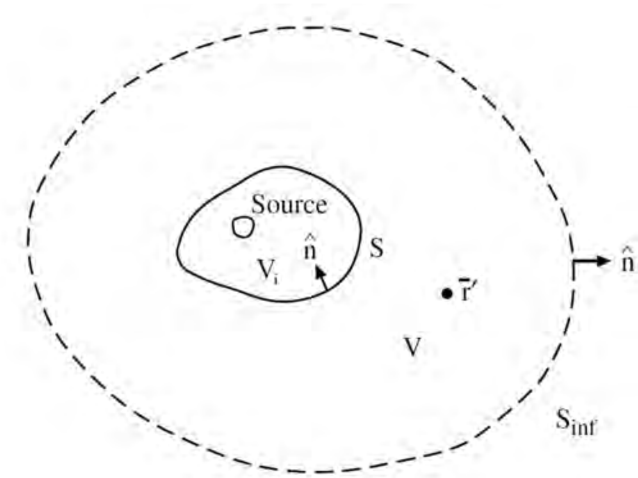

Figure 31.7: The geometry for deriving Huygens' principle. The radiation from the source can be replaced by equivalence sources on the surface S, and the field outside S can be calculated using (31.4.4).

If the volume V is bounded by S and S_{inf} as shown in Figure 31.7, then the surface integral in (31.4.4) should include an integral over S_{inf}. But when $S_{\text{inf}} \to \infty$, all fields look like plane wave, and $\nabla \to -\hat{r}jk$ on S_{inf}. Furthermore, $g(\mathbf{r}-\mathbf{r}') \sim O(1/r)$,[4] when $r \to \infty$, and $\psi(\mathbf{r}) \sim O(1/r)$, when $r \to \infty$, if $\psi(\mathbf{r})$ is due to a source of finite extent. Then, the integral over S_{inf} in (31.4.4) vanishes, and (31.4.4) is valid for the case shown in Figure 31.7 as well but with the surface integral over surface S only. Here, the field outside S at $\mathbf{r}'$ is expressible in terms of the field on S. This is similar to the inside-out equivalence principle we have discussed previously.

Notice that in deriving (31.4.4), $g(\mathbf{r}, \mathbf{r}')$ has only to satisfy (31.4.2) for both $\mathbf{r}$ and $\mathbf{r}'$ in V but no boundary condition has yet been imposed on $g(\mathbf{r}, \mathbf{r}')$. Therefore, if we further require

[4]The symbol "O" means "of the order."

that $g(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r} \in S$, then (31.4.4) becomes

$$\psi(\mathbf{r}') = -\oint_S dS\, \psi(\mathbf{r})\, \hat{n} \cdot \nabla g(\mathbf{r}, \mathbf{r}'), \qquad \mathbf{r}' \in V. \tag{31.4.5}$$

On the other hand, if require additionally that $g(\mathbf{r}, \mathbf{r}')$ satisfies (31.4.2) with the boundary condition $\hat{n} \cdot \nabla g(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r} \in S$, then (31.4.4) becomes

$$\psi(\mathbf{r}') = \oint_S dS\, g(\mathbf{r}, \mathbf{r}')\, \hat{n} \cdot \nabla \psi(\mathbf{r}), \qquad \mathbf{r}' \in V. \tag{31.4.6}$$

Equations (31.4.4), (31.4.5), and (31.4.6) are various forms of Huygens' principle, or equivalence principle for scalar waves (acoustic waves) depending on the definition of $g(\mathbf{r}, \mathbf{r}')$. Equations (31.4.5) and (31.4.6) stipulate that only $\psi(\mathbf{r})$ or $\hat{n} \cdot \nabla \psi(\mathbf{r})$ need be known on the surface S in order to determine $\psi(\mathbf{r}')$. The above are analogous to the PEC and PMC equivalence principle considered previously. (Note that in the above derivation, k^2 could be a function of position as well.)

31.4.2 Electromagnetic Waves Case

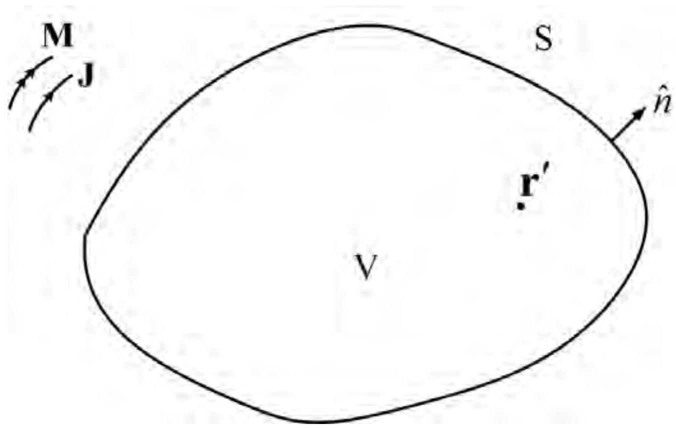

Figure 31.8: Derivation of the Huygens' principle for the electromagnetic case. One only needs to know the surface fields on surface S in order to determine the field at $\mathbf{r}'$ inside V.

The derivation of Huygens' principle and Green's theorem for the electromagnetic case is more complicated than the scalar wave case. But fortunately, this problem is mathematically homomorphic to the scalar wave case. In dealing with the requisite vector algebra, we have to remember to cross the t's and dot the i's, to carry ourselves through the laborious vector algebra.

In a source-free region, an electromagnetic wave satisfies the vector wave equation

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\, \mathbf{E}(\mathbf{r}) = 0. \tag{31.4.7}$$

The analogue of the scalar Green's function for the scalar wave equation is the dyadic Green's function for the electromagnetic wave case [1, 31, 172, 173]. Moreover, the dyadic Green's function satisfies the equation[5]

$$\nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - k^2\, \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \bar{\mathbf{I}}\, \delta(\mathbf{r} - \mathbf{r}'). \tag{31.4.8}$$

It can be shown by direct back substitution that the dyadic Green's function in free space is [173]

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) g(\mathbf{r} - \mathbf{r}') \tag{31.4.9}$$

The above allows us to derive the vector Green's theorem [1, 31, 172].

Then, after post-multiplying (31.4.7) by $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$, pre-multiplying (31.4.8) by $\mathbf{E}(\mathbf{r})$, subtracting the resultant equations and integrating the difference over volume V, considering two cases as we did for the scalar wave case, we have

$$\left.\begin{array}{ll}\text{if } \mathbf{r}' \in V, & \mathbf{E}(\mathbf{r}') \\ \text{if } \mathbf{r}' \notin V, & 0\end{array}\right\} = \int_V dV\, \left[\mathbf{E}(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \right. \\ \left. - \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')\right]. \tag{31.4.10}$$

Next, using the vector identity that[6]

$$-\nabla \cdot \left[\mathbf{E}(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') + \nabla \times \mathbf{E}(\mathbf{r}) \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')\right] \\ = \mathbf{E}(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}'), \tag{31.4.11}$$

Equation (31.4.10), with the help of Gauss' divergence theorem, can be written as

$$\left.\begin{array}{ll}\text{if } \mathbf{r}' \in V, & \mathbf{E}(\mathbf{r}') \\ \text{if } \mathbf{r}' \notin V, & 0\end{array}\right\} = -\oint_S dS\, \hat{n} \cdot \left[\mathbf{E}(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') + \nabla \times \mathbf{E}(\mathbf{r}) \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')\right] \\ = -\oint_S dS\, \left[\hat{n} \times \mathbf{E}(\mathbf{r}) \cdot \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') + i\omega\mu\, \hat{n} \times \mathbf{H}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')\right]. \tag{31.4.12}$$

The above is just the vector analogue of (31.4.4). We have used the cyclic relation of dot and cross products to rewrite the last expression. Since $\mathbf{E} \times \hat{n}$ and $\hat{n} \times \mathbf{H}$ are associated with

[5] A dyad is an outer product between two vectors, and it behaves like a tensor, except that a tensor is more general than a dyad. A purist will call the above a tensor Green's function, as the above is not a dyad in its strictest definition.

[6] This identity can be established by using the identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$. We will have to let (31.4.11) act on a constant vector to convert the dyad into a vector before applying this identity. The equality of the volume integral in (31.4.10) to the surface integral in (31.4.12) is also known as the vector Green's theorem [31, 172]. Earlier form of this theorem was known as Franz formula [174].

surface magnetic current and surface electric current, respectively, the above can be thought of having these equivalence surface currents radiating via the dyadic Green's function. Again, notice that (31.4.12) is derived via the use of (31.4.8), but no boundary condition has yet been imposed on $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ on S even though we have given a closed form solution for the free-space case.

Now, if we require the additional boundary condition that $\hat{n}\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')=0$ for $\mathbf{r}\in S$. This corresponds to a point source radiating in the presence of a PEC surface. Then (31.4.12) becomes

$$\mathbf{E}(\mathbf{r}') = -\oint_S dS\,\hat{n}\times\mathbf{E}(\mathbf{r})\cdot\nabla\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}'), \qquad \mathbf{r}'\in V \tag{31.4.13}$$

for it could be shown that $\hat{n}\times\mathbf{H}\cdot\overline{\mathbf{G}}=\mathbf{H}\cdot\hat{n}\times\overline{\mathbf{G}}$ implying that the second term in (31.4.12) is zero. On the other hand, if we require that $\hat{n}\times\nabla\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')=0$ for $\mathbf{r}\in S$, then (31.4.12) becomes

$$\mathbf{E}(\mathbf{r}') = -i\omega\mu\oint_S dS\,\hat{n}\times\mathbf{H}(\mathbf{r})\cdot\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}'), \qquad \mathbf{r}'\in V \tag{31.4.14}$$

Equations (31.4.13) and (31.4.14) state that $\mathbf{E}(\mathbf{r}')$ is determined if either $\hat{n}\times\mathbf{E}(\mathbf{r})$ or $\hat{n}\times\mathbf{H}(\mathbf{r})$ is specified on S. This is in agreement with the uniqueness theorem. These are the mathematical expressions of the PEC and PMC equivalence problems we have considered in the previous sections.

The dyadic Green's functions in (31.4.13) and (31.4.14) are for a closed cavity since boundary conditions are imposed on S for them. But the dyadic Green's function for an unbounded, homogeneous medium, given in (31.4.10) can be written as

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \frac{1}{k^2}[\nabla\times\nabla\times\overline{\mathbf{I}}\,g(\mathbf{r}-\mathbf{r}')-\overline{\mathbf{I}}\,\delta(\mathbf{r}-\mathbf{r}')], \tag{31.4.15}$$

$$\nabla\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \nabla\times\overline{\mathbf{I}}\,g(\mathbf{r}-\mathbf{r}'). \tag{31.4.16}$$

Then, (31.4.12), for $\mathbf{r}'\in V$ and $\mathbf{r}'\neq\mathbf{r}$, becomes

$$\mathbf{E}(\mathbf{r}') = -\nabla'\times\oint_S dS\,g(\mathbf{r}-\mathbf{r}')\,\hat{n}\times\mathbf{E}(\mathbf{r}) + \frac{1}{i\omega\epsilon}\nabla'\times\nabla'\times\oint_S dS\,g(\mathbf{r}-\mathbf{r}')\,\hat{n}\times\mathbf{H}(\mathbf{r}). \tag{31.4.17}$$

The above can be applied to the geometry in Figure 31.7 where $\mathbf{r}'$ is enclosed in S and S_{inf}. However, the integral over S_{inf} vanishes by virtue of the radiation condition as for (31.4.4). Then, (31.4.17) relates the field outside S at $\mathbf{r}'$ in terms of only the field on S. This is similar to the inside-out problem in the equivalence theorem. It is also related to the fact that if the radiation condition is satisfied, then the field outside of the source region is uniquely satisfied. Hence, this is also related to the uniqueness theorem.